The Fisher-information-based uncertainty relation, Cramer–Rao inequality and kinetic energy

for the D-dimensional central problem

The Fisher-information-based uncertainty relation, Cramer–Rao inequality and kinetic energy for the *D*-dimensional central problem

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Abstract

The inequality $\langle p^2 \rangle \geqslant \left(L + \frac{1}{2}\right)^2 \langle r^{-2} \rangle$, with L being the grand orbital quantum number, and its conjugate relation for $(\langle r^2 \rangle, \langle p^{-2} \rangle)$ are shown to be fulfilled in the D-dimensional central problem. Their use has allowed us to improve the Fisher-information-based uncertainty relation ($I_\rho I_\gamma \geqslant$ const) and the Cramer–Rao inequalities ($\langle r^2 \rangle I_\rho \geqslant D^2$; $\langle p^2 \rangle I_\gamma \geqslant D^2$). In addition, the kinetic energy and the radial expectation value $\langle r^2 \rangle$ are shown to be bounded from below by the Fisher information in position and momentum spaces, denoted by I_ρ and I_γ, respectively.

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1. Introduction

The essential inadequacy of the classical position and momentum concepts for a single particle in a D-dimensional physical system is quantum mechanically shown by the celebrated variance-based Heisenberg relation [1, 2] and its moment generalizations [3–6]. As well, this can be done in a much more appropriate and stringent manner by other position–momentum uncertainty relations which use information-theoretic quantities of global type as uncertainty measures: the entropic or Shannon-entropy-based [7, 8], Renyi-entropy-based [3, 9] and Tsallis-entropy-based [10] inequalities.

A qualitatively different position–momentum uncertainty relation has been recently suggested [11–13] but not yet set up for general systems. In contrast to the previous inequalities, it is based on gradient-like uncertainty measures (so, of local, as opposed to global type): the Fisher informations [14–16], I_ρ and I_γ, of the single-particle position and

momentum densities $\rho(\vec{r})$ and $\gamma(\vec{p})$, respectively. The position Fisher information I_ρ is given by

$$I_\rho = \int_{\mathbb{R}^D} \frac{|\vec{\nabla}_D \rho(\vec{r})|^2}{\rho(\vec{r})} \, \mathrm{d}^D r = 4 \int_{\mathbb{R}^D} |\vec{\nabla}_D \sqrt{\rho(\vec{r})}|^2 \, \mathrm{d}^D r.$$

This shift-invariance Fisher quantity, which increases with the concentration of the single-particle density, is a very fertile concept in large part because of its flexibility and multiple meanings [16]. It is called the 'Fisher channel capacity' [16], although for brevity we simply call it 'Fisher information' here. It does not only measure the position uncertainty of the particle but it is closely connected to a wide variety of physical quantities (e.g. kinetic [16–19] and Weiszäcker [16, 18, 20] energies) and it has been used to understand numerous quantum-mechanical phenomena such as the spectral avoided-crossings of atoms in strong external fields [21] and the correlation properties of two-electron systems [22, 23]. Moreover, the Fisher information is the basic variable of the principle of extreme physical information [16, 23–26] which has been used to obtain various fundamental equations of motion in physics [16, 26]. It is also being used to rederive classical thermodynamics without the usual concept of Boltzmann's entropy [27, 28].

Finally, let us mention that this information-theoretic quantity and its quantum extension [29], not yet sufficiently well known for physicists, has been used to set up a number of relevant inequalities (such as, e.g., Cramer–Rao [20, 29–34] and uncertainty relations [11, 13, 29, 31, 34–36]). In D-dimensional physics, the Fisher information of single-particle systems has been only recently determined in closed form in terms of the quantum numbers characterizing the involved physical state for both position and momentum spaces [37].

The new uncertainty relation has the form

$$I_\rho I_\gamma \geqslant K(D), \tag{1}$$

where the $K(D)$ is a constant to be determined. It is known that $K(1) = 4$ for general monodimensional systems for even wavefunctions [12]. Moreover, the uncertainty character of the product of the position and momentum Fisher distributions is shown by [35]

$$I_\rho I_\gamma \geqslant 16 \left[1 - \frac{(2L+1)|m|}{L(L+1) - \frac{1}{2}(D-1)(D-3)} \right]^2 \langle r^2 \rangle \langle p^2 \rangle, \tag{2}$$

for central potentials, where the grand orbital quantum number is $L = l + (D-3)/2$, and (l, m) are the orbital and magnetic quantum numbers.

Then, the consideration of the modified Heisenberg inequality $\langle r^2 \rangle \langle p^2 \rangle \geqslant D^2/4$ (valid for general systems) [38–41] into (2) have led to the Fisher-product lower bound [35]

$$I_\rho I_\gamma \geqslant 4D^2 \left[1 - \frac{(2L+1)|m|}{L(L+1) - \frac{1}{2}(D-1)(D-3)} \right]^2 \equiv K_1(l, m; D). \tag{3}$$

The lower bound $D^2/4$ to the Heisenberg product has been just refined for central potentials [13] as

$$\langle r^2 \rangle \langle p^2 \rangle \geqslant \left(L + \tfrac{3}{2} \right)^2, \tag{4}$$

so that it, together with (2), produces the following improved value for $K(D)$ [13]:

$$I_\rho I_\gamma \geqslant 16 \left[1 - \frac{(2L+1)|m|}{L(L+1) - \frac{1}{2}(D-1)(D-3)} \right]^2 \left(L + \frac{3}{2} \right)^2$$

$$\equiv K_2(l, m; D). \tag{5}$$

On the other hand, there exist two other relevant inequalities which involve the Fisher information in a given space and a radial expectation value in the same (Cramer–Rao) or the conjugate (Stam) space. They are the Stam uncertainty relations [6, 30, 42]

$$I_\rho \leqslant 4\langle p^2\rangle; \qquad I_\gamma \leqslant 4\langle r^2\rangle, \tag{6}$$

and the Cramer–Rao inequalities [20, 30]

$$\langle r^2\rangle I_\rho \geqslant D^2; \qquad \langle p^2\rangle I_\gamma \geqslant D^2. \tag{7}$$

There are three main achievements in this paper for *D*-dimensional single-particle systems with central potentials: to further improve the lower bound $K_2(l, m; D)$ given by (5) for the Fisher uncertainty product $I_\rho I_\gamma$, to refine the Cramer–Rao inequalities and to find lower bounds to the kinetic energy (which together with the Stam inequality allows one to bound this quantity in both senses) and to the radial expectation value $\langle r^2\rangle$. These results have been possible because we have improved the radial-uncertainty-like inequalities which involve the radial expectation values $(\langle p^2\rangle, \langle r^{-2}\rangle)$ and $(\langle r^2\rangle, \langle p^{-2}\rangle)$.

The structure of the paper is the following. First, in section 2, the wave equation of a particle in a central potential is briefly discussed in *D*-dimensional configuration space, and some useful notions and notation used throughout the paper are given. Then, in section 3, the basic radial-uncertainty-like inequalities are derived. The uncertainty relation based on the Fisher informations in position and momentum spaces is refined in section 4. We express the improvement of the Cramer–Rao inequalities and the bounds to the kinetic energy and the radial expectation value $\langle r^2\rangle$ in section 5. Then, we devote section 6 to a numerical study of the Fisher-information-based uncertainty relation for two illustrative examples: the harmonic oscillator and the hydrogen atom. Finally, some conclusions are given. Atomic units ($e = \hbar = m_e = 1$) are used throughout the paper.

2. *D*-dimensional central potentials

The Schrödinger equation of a *D*-dimensional particle in the central potential $V_D(r)$ is

$$\left[-\tfrac{1}{2}\vec{\nabla}_D^2 + V_D(r)\right]\psi_D(\vec{r}) = E_D\psi_D(\vec{r}),$$

where the position vector $\vec{r}$ has the *D* polar hyperspherical coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{D-1}) \equiv (r, \Omega_{D-1})$, and $\vec{\nabla}_D^2$ denotes the Laplacian operator. See [43–47] for a detailed analysis of the *D*-dimensional problem for central potentials and [13, 35] for a better understanding of the notation and definitions used in this work. Let us here describe the wavefunctions of the problem as

$$\psi_{E,l,\{\mu\}}(\vec{r}) = R_{E,l}(r)\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1}), \tag{8}$$

where $\mathcal{Y}$-functions denote the hyperspherical harmonics characterized by the $D-1$ hyperangular quantum numbers $(l \equiv \mu_1, \mu_2, \ldots, \mu_{D-1}) \equiv (l, \{\mu\})$, which are natural numbers with values $l = 0, 1, 2, \ldots$, and $l \equiv \mu_1 \geqslant \mu_2 \geqslant \cdots \mu_{D-2} \geqslant |\mu_{D-1}| \equiv |m|$. These hyperspherical harmonics are the eigenfunctions of the squared hyperangular momentum operator corresponding to the eigenvalues $l(l + D - 2) = L(L + 1) - (D - 1)(D - 3)/4$.

The radial wavefunctions $R_{El}(r)$ are the solutions of the equation

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} - \frac{D-1}{r}\frac{d}{dr} + \frac{l(l + D - 2)}{2r^2} + V_D(r)\right]R_{E,l}(r) = E_D R_{E,l}(r).$$

This equation transforms into the reduced radial Schrödinger equation

$$\left[-\frac{1}{2}\frac{d^2}{dr^2} + \frac{L(L+1)}{2r^2} + V_D(r)\right]u_{E,l}(r) = E_D u_{E,l}(r)$$

by means of the change

$$R_{E,l} \to u_{E,l} : u_{E,l} = r^{(D-1)/2} R_{E,l}(r). \tag{9}$$

The normalization to unity of the wavefunctions imposes that the reduced radial wavefunctions $u_{E,l}(r)$ are normalized as

$$\int_0^\infty u_{E,l}^2(r)\, dr = 1,$$

once we take into account the known orthonormalization relations of the hyperspherical harmonics [43, 45, 47].

The probability density of the D-dimensional particle in position space is given by

$$\begin{aligned} \rho_{E,l,\{\mu\}}(\vec{r}) &= |\psi_{E,l,\{\mu\}}(\vec{r})|^2 = R_{E,l}^2(r)|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2 \\ &= r^{1-D} u_{E,l}^2(r)|\mathcal{Y}_{l,\{\mu\}}(\Omega_{D-1})|^2. \end{aligned} \tag{10}$$

The spreading of this density all over the D-dimensional space is usually quantified by means of a radial expectation value

$$\langle f(r)\rangle = \int_{\mathbb{R}^D} f(r)\rho_{E,l,\{\mu\}}(\vec{r})\, d\vec{r} = \int_0^\infty f(r)u_{E,l}^2(r)\, dr$$

and, more appropriately, by the use of an information-theoretic quantity of global (Renyi, Shannon) and local (Fisher) types. The probability density of the particle in momentum space $\gamma_{E,l,\{\mu\}}(\vec{p}) = |\tilde{\psi}_{E,l,\{\mu\}}(\vec{p})|^2$, $\tilde{\psi}$ denoting the Fourier transform of ψ, can be separated out in radial and angular parts similarly as for the position density following (8) and (10).

3. Radial-uncertainty-like inequalities

Here, we shall derive the conjugate uncertainty inequalities which link the radial expectation values ($\langle p^2\rangle$, $\langle r^{-2}\rangle$) and ($\langle r^2\rangle$, $\langle p^{-2}\rangle$). We know [35] that the momentum expectation value $\langle p^2\rangle$ can be decomposed for central potentials in the form

$$\langle p^2\rangle = J_R(D) + \left[L(L+1) - \tfrac{1}{4}(D-1)(D-3)\right]\langle r^{-2}\rangle, \tag{11}$$

with the radial integral

$$J_R(D) = \int_0^\infty \left[\frac{dR_{E,l}(r)}{dr}\right]^2 r^{D-1}\, dr. \tag{12}$$

The non-negativity of this integral straightforwardly leads [35] to the following radial-uncertainty-like relation:

$$\langle p^2\rangle \geqslant \left[L(L+1) - \tfrac{1}{4}(D-1)(D-3)\right]\langle r^{-2}\rangle. \tag{13}$$

The corresponding inequality in momentum space, following a parallel procedure, is [35]

$$\langle r^2\rangle \geqslant \left[L(L+1) - \tfrac{1}{4}(D-1)(D-3)\right]\langle p^{-2}\rangle. \tag{14}$$

The two expressions above can be improved by taking into account the change (9) in the radial integral (12). Then, one obtains

$$\begin{aligned} J_R(D) &= \int_0^\infty \left[(u')^2 + \left(\frac{D-1}{2}\right)^2 r^{-2}u^2 - (D-1)r^{-1}uu'\right] dr \\ &= \int_0^\infty (u')^2\, dr + \frac{1}{4}(D-1)(D-3)\langle r^{-2}\rangle, \end{aligned} \tag{15}$$

where we have used the equality

$$\int_0^\infty r^{-1}uu' \, dr = \frac{1}{2} \int_0^\infty r^{-2}u^2 \, dr$$

provided that $u(r) \sim r^{L+1}$ at $r = 0$ (which implies that $D > 2$), which occurs for any central potential satisfying $|V(r)| \leqslant Cr^{-2}$ at $r \to 0$.

Then, the momentum expectation value given by (11) together with (15) transforms into

$$\langle p^2 \rangle = \int_0^\infty (u')^2 \, dr + L(L+1)\langle r^{-2} \rangle,$$

which, due to the non-negativity of the integral, leads to the inequality

$$\langle p^2 \rangle \geqslant L(L+1)\langle r^{-2} \rangle. \tag{16}$$

A similar procedure in momentum space allows us to obtain

$$\langle r^2 \rangle \geqslant L(L+1)\langle p^{-2} \rangle. \tag{17}$$

These two radial-uncertainty-like relations improve the corresponding inequalities (13) and (14). A further refinement of these two relations (16) and (17) can be obtained from the inequality

$$\int_0^\infty \left(u' - \frac{\lambda}{r}u \right)^2 \, dr \geqslant 0,$$

with the real parameter λ. Working out this integral one has the λ-inequality

$$\langle r^{-2} \rangle \lambda^2 - \langle r^{-2} \rangle \lambda + \langle p^2 \rangle - L(L+1)\langle r^{-2} \rangle \geqslant 0,$$

whose negative discriminant gives rise to the inequality

$$\langle p^2 \rangle \geqslant \left(L + \tfrac{1}{2} \right)^2 \langle r^{-2} \rangle. \tag{18}$$

The corresponding conjugate relation is

$$\langle r^2 \rangle \geqslant \left(L + \tfrac{1}{2} \right)^2 \langle p^{-2} \rangle. \tag{19}$$

The last two radial-uncertainty-like inequalities improve for central potentials the corresponding general Faris [48] and Pitt–Beckner [42, 49, 50] inequalities, and the similar central potential lower bounds recently found [13, 35] as well as inequalities (16) and (17). In addition, they extend similar expressions found by other authors [19, 38, 39, 41].

4. Fisher-information-based uncertainty relation

Here, we shall refine the Fisher-information-based uncertainty relation given by (5). Recently, it has been found [35] that the position and momentum Fisher informations I_ρ and I_γ, respectively, can be expressed in terms of the radial expectation values $\langle r^k \rangle$ and $\langle p^k \rangle$, $k = -2$ and 2, as

$$I_\rho = 4\langle p^2 \rangle - 2(2L+1)|m|\langle r^{-2} \rangle, \qquad I_\gamma = 4\langle r^2 \rangle - 2(2L+1)|m|\langle p^{-2} \rangle.$$

The combination of these two exact expressions with the radial-uncertainty-like relations (18) and (19) allows us to obtain

$$I_\rho \geqslant 4 \left(1 - \frac{2|m|}{2L+1} \right) \langle p^2 \rangle, \tag{20}$$

$$I_\gamma \geqslant 4 \left(1 - \frac{2|m|}{2L+1} \right) \langle r^2 \rangle. \tag{21}$$

The multiplication of (20) and (21) leads to the following relationship between the Fisher and Heisenberg uncertainty products:

$$I_\rho I_\gamma \geqslant 16 \left(1 - \frac{2|m|}{2L+1}\right)^2 \langle r^2 \rangle \langle p^2 \rangle,$$

which improves the corresponding relation (2).

Then, taking into account the modified Heisenberg relation for central potentials given by (4) one finally has the Fisher-information-based uncertainty relation

$$I_\rho I_\gamma \geqslant 16 \left(1 - \frac{2|m|}{2L+1}\right)^2 \left(L + \frac{3}{2}\right)^2 \equiv K_3(l, m; D), \tag{22}$$

which further improves the inequality (5) since $K_3 \geqslant K_2$.

According to (22), the lower bound to the Fisher product $I_\rho I_\gamma$ is equal to $4D^2$ for states s (i.e. with $l = 0$) and to $16(l + D/2)^2$ for levels with $m = 0$. Moreover, for a three-dimensional single-particle system one has

$$I_\rho I_\gamma \geqslant 16 \left(1 - \frac{2|m|}{2l+1}\right)^2 \left(l + \frac{3}{2}\right)^2,$$

so that $I_\rho I_\gamma \geqslant 36$ for its ground state ($l = m = 0$).

5. Cramer–Rao inequality and kinetic energy bounds

Here, we will improve for central potentials the general Cramer–Rao inequalities (7) and then we will obtain lower bounds for the position and momentum Fisher informations in terms of $\langle p^2 \rangle$ and $\langle r^2 \rangle$, respectively. The latter provide us with lower bounds to the kinetic energy and the radial expectation value $\langle r^2 \rangle$ in terms of these local information-theoretic quantities.

The Cramer–Rao inequality in position space can be, according to (20), bounded from below as

$$\langle r^2 \rangle I_\rho \geqslant 4 \left(1 - \frac{2|m|}{2L+1}\right) \langle r^2 \rangle \langle p^2 \rangle.$$

Then, taking into account the D-dimensional Heisenberg relation (4) one has

$$\langle r^2 \rangle I_\rho \geqslant 4 \left(1 - \frac{2|m|}{2L+1}\right) \left(L + \frac{3}{2}\right)^2. \tag{23}$$

In a similar manner, according to (21) and (4), the Cramer–Rao inequality in momentum space is

$$\langle p^2 \rangle I_\gamma \geqslant 4 \left(1 - \frac{2|m|}{2L+1}\right) \left(L + \frac{3}{2}\right)^2. \tag{24}$$

It is worth pointing out that the new lower bounds to both position and momentum Cramer–Rao products (i) are equal to D^2 for states s (i.e. with $l = 0$) and (ii) substantially improve the D^2-value given by (7) for general systems.

The expressions (20) and (21) together with the Stam uncertainty relations (6) allow us to bound the kinetic energy $T (= \langle p^2 \rangle / 2)$ in both senses as

$$\frac{1}{8} I_\rho \leqslant T \leqslant \frac{1}{8} \frac{2L+1}{2L+1-2|m|} I_\rho \tag{25}$$

in terms of the position Fisher information, and the radial expectation value $\langle r^2 \rangle$ (which is closely connected to numerous physical quantities such as, e.g., the Langevin–Pauli
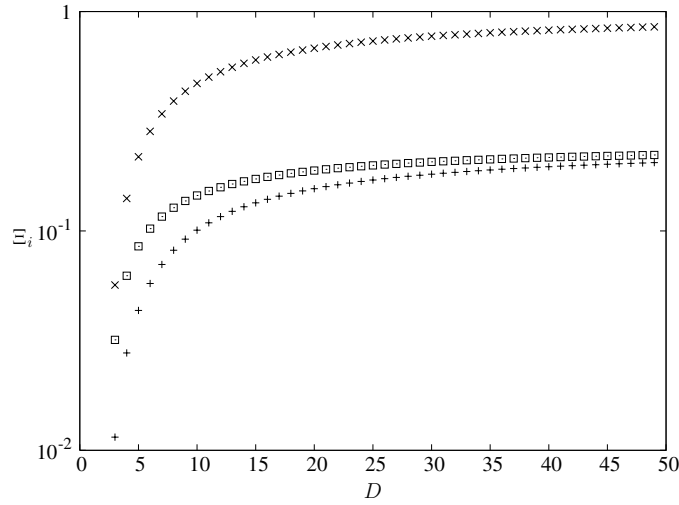
Figure 1. Ratios Ξ_i, $i = 1(+)$, $2(\square)$ and $3(\times)$, of the lower bounds and the Fisher product in terms of the dimension D for the state with quantum numbers $(n, l, m) = (1, 1, 1)$ of the harmonic oscillator.

diamagnetic susceptibility [51] $\chi = -\alpha^2 \langle r^2 \rangle / 6$, with α being the fine structure constant) as

$$\frac{1}{4} I_\gamma \leqslant \langle r^2 \rangle \leqslant \frac{1}{4} \frac{2L+1}{2L+1-2|m|} I_\gamma \qquad (26)$$

in terms of the momentum Fisher information.

6. Numerical study of the Fisher-information inequality

In this section, we have performed a comparison between the three Fisher-information-based uncertainty relations (3), (5) and (22), in the two most important prototypes of *D*-dimensional systems: the isotropic harmonic oscillator and the hydrogen atom. To facilitate our discussion, let us introduce the ratios between each of the associated lower bounds and the corresponding Fisher product $I_\rho I_\gamma$:

$$\Xi_i = \frac{K_i}{I_\rho I_\gamma} \leqslant 1 \qquad i = 1, 2, 3.$$

where $K_i \equiv K_i(l, m; D)$ are the lower bounds given by (3), (5) and (22), respectively. We have numerically studied the dependence of these ratios on the dimensionality and on the principal quantum number of the oscillator and hydrogen-like states.

6.1. Isotropic harmonic oscillator

The expressions of the Fisher informations in position and momentum spaces for the oscillator potential $V(r) = \frac{1}{2}\omega^2 r^2$ (mass = 1) are given by [35]

$$I_\rho = 4 \left(2n + l - |m| + \frac{D}{2} \right) \omega \qquad I_\gamma = 4 \left(2n + l - |m| + \frac{D}{2} \right) \omega^{-1}.$$

The behaviour of the ratio Ξ_i ($i = 1, 2$ and 3) as a function of the dimension D for the state with quantum numbers $(n, l, m) = (1, 1, 1)$ is shown in figure 1. For the three cases,
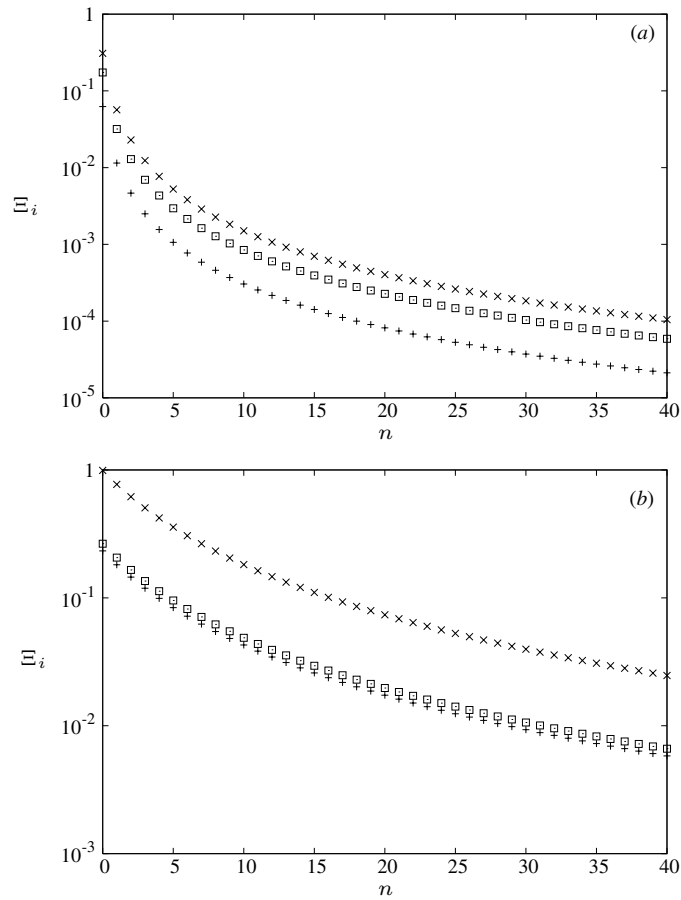
Figure 2. Ratios Ξ_i, $i = 1(+), 2(\square)$ and $3(\times)$, of the lower bounds and the Fisher product for the harmonic oscillator state with $l = m = 1$ in terms of the principal quantum number n for dimensions $D = 3$ (*a*) and $D = 30$ (*b*).

the ratio monotonically increases, i.e. the bounds improve, as D is augmented. For any given dimensionality our results, given by (22), provide a significant improvement of the uncertainty relations (3) and (5). Indeed Ξ_3 is much closer to the unity than Ξ_1 and Ξ_2 and it holds $\Xi_3 \rightarrow 1$ in the $D \rightarrow \infty$ limit. The asymptotic trend is very different for the previously found bounds; and, for a fixed set of quantum numbers (n, l, m), both satisfy $\Xi_i \rightarrow \left(\frac{2l - |m|}{2l}\right)^2$ for $D \rightarrow \infty$, with $i = 1$ and 2. Note that only for levels with a vanishing magnetic quantum number $\Xi_{1,2} \rightarrow 1$ for $D \rightarrow \infty$, while for the state $(1, 1, 1)$ analysed here $\Xi_{1,2} \rightarrow \frac{1}{4}$ for large D values.

Figures 2(*a*) and (*b*) present the ratio Ξ_i ($i = 1, 2$ and 3) for the states with rotational and magnetic quantum numbers $l = m = 1$, as a function of the principal quantum number n for $D = 3$ and $D = 30$, respectively. Fixing the angular symmetry of the states (i.e. l and m) and the dimension D, the values of the bounds K_1, K_2 and K_3 are also fixed, and only the product $I_\rho I_\gamma$ varies as n is increased. For the three cases, Ξ_i has a qualitatively similar but quantitatively different behaviour as a function of n, monotonically decreasing as n is enhanced. The inequalities worsen as the degree of excitation of the level is augmented, and
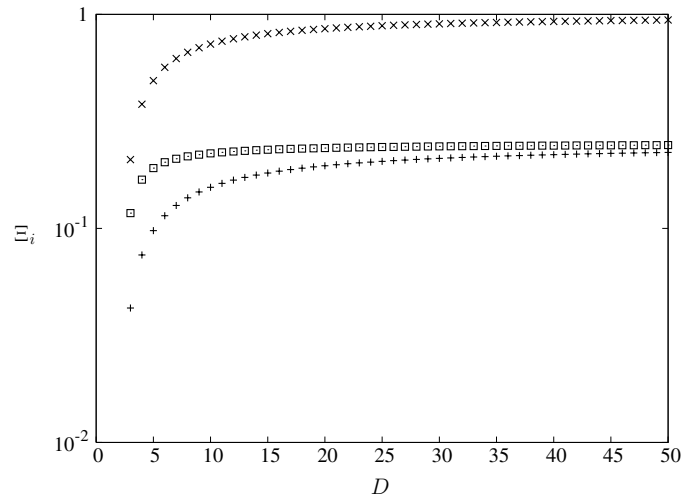
Figure 3. Ratios $\Xi_i, i = 1(+), 2(\square)$ and $3(\times)$, of the lower bounds and the Fisher product in terms of the dimension D for the state with quantum numbers $(n, l, m) = (2, 1, 1)$ of the hydrogen atom.

in the $n \to \infty$ limit $\Xi_i \to 0$ for $i = 1, 2$ and 3. For $D = 3$, our new expression (22) shows a less improvement with respect to the previous ones, as it is seen in figure 2(*a*). On the other hand, the ratio Ξ_3 is reduced in more than three orders of magnitude for the considered states, from $\Xi_3 = 0.31$ to 0.1×10^{-3} for the levels with $n = 0$ and $n = 40$, respectively. However, for $D = 30$ the uncertainty relation (22) represents a significant enhancement with respect to the previous results (3) and (5), as it is illustrated by figure 2(*b*). Indeed, this inequality almost saturate, with $\Xi_3 = 0.99$ for the level with $n = 0$, decreasing thereafter. For the set of considered states Ξ_3 is reduced by more than one order of magnitude, the highest excited state considered with $n = 40$ has $\Xi_3 = 0.025$.

6.2. Hydrogen atom

The expressions of the Fisher informations in position and momentum spaces for the hydrogenic potential, $V(r) = -1/r$, read [35]

$$I_\rho = \frac{4}{\eta^3}(\eta - |m|) \qquad I_\gamma = 2\eta^2\{5\eta^2 - 3L(L+1) - [8\eta - 3(2L+1)]|m| + 1\},$$

with $\eta = n + (D - 3)/2$

Figure 3 shows the behaviour of the ratio Ξ_i ($i = 1, 2$ and 3) as a function of the dimension D for the level with quantum numbers $(n, l, m) = (2, 1, 1)$ of the hydrogen atom. These results resemble those presented for the harmonic oscillator, see figure 1. The three lower bounds increase as the dimensionality of the system is enhanced. The improvement provided by our results is clearly manifested in this graphic; indeed, Ξ_3 quickly reaches the asymptotic behaviour for large D values, with $\Xi_3 \to 1$ for $D \to \infty$, i.e. the inequality (22) saturates. The ratios Ξ_1 and Ξ_2 also approach $\left(\frac{2l - |m|}{2l}\right)^2$ on the $D \to \infty$ limit, which is equal to $\frac{1}{4}$ for this state, being well below Ξ_3.

As a last example, we present for the hydrogenic levels with angular symmetry $m = l = 1$ the ratios Ξ_i, with $i = 1, 2$ and 3, as a function of the principal quantum number n for a
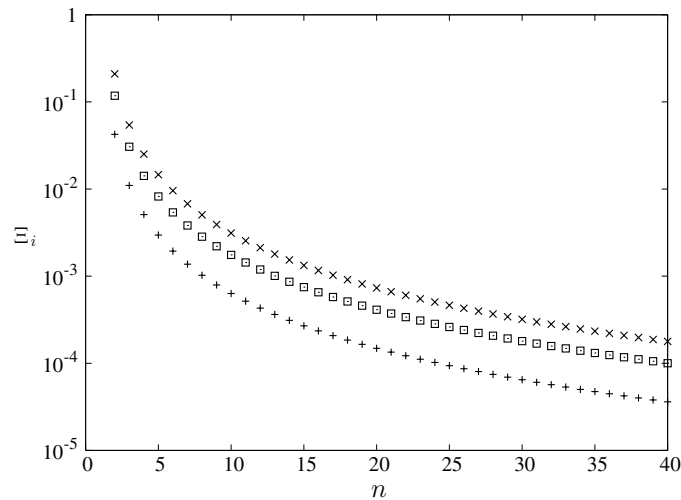
Figure 4. Ratios Ξ_i, $i = 1(+)$, $2(\square)$ and $3(\times)$, of the lower bounds and the Fisher product for the hydrogenic state with $l = m = 1$ in terms of the principal quantum number n for dimension $D = 3$.

three-dimensional system in figure 4. As for the oscillator case, the three bounds worsen as the degree of excitation is enhanced and they approach zero in the $n \to \infty$ limit. K_3 slightly improves the results given by K_1 and K_2. Again, Ξ_3 decreases in more than three orders of magnitude for the considered levels, from $\Xi_3 = 0.21$ to 0.18×10^{-3} for the states with $n = 0$ and $n = 40$, respectively.

7. Conclusions

The Fisher-information-based uncertainty relation and the Cramer–Rao inequality for D-dimensional particles moving in arbitrary central potentials have been best set up, see (22) and (23)–(24), respectively. The lower bounds to the associated Fisher and Cramer–Rao products are given by means of the orbital and magnetic hyperangular quantum numbers in a simple and closed form. This has been possible because of the improvement of the Faris–Pitt–Beckner inequalities for central potentials. On the other hand, the kinetic energy and the radial expectation value $\langle r^2 \rangle$ are shown to be bounded not only from above (which is known due to the Stam uncertainty relation) but also from below in terms of the position and momentum Fisher informations and the hyperangular quantum numbers already mentioned, see (25) and (26), respectively.

In addition, we have numerically investigated the accuracy of the Fisher-information-based uncertainty relation (22) for the two most prominent D-dimensional prototypes: the isotopic harmonic oscillator and hydrogen atom. Moreover, we have compared it to the previously known results given by expressions (3) and (5). For all considered physical situations, the new lower bound K_3 is systematically better than K_1 and K_2, the improvement factor being strongly dependent on the symmetry of the selected state and on the dimensionality of the system. For all levels our inequality (22) saturates on the $D \to \infty$ limit, i.e. $K_3 \to I_\rho I_\gamma$, while the asymptotic behaviour of the previous bounds strongly depends on the state under consideration. The saturation is only attained for $m = 0$ levels. For a given dimension, with the quantum numbers l and m fixed, the three bounds satisfy $K_i \to 0$ with $i = 1, 2$ and 3 on the $n \to \infty$ limit.

Acknowledgments

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